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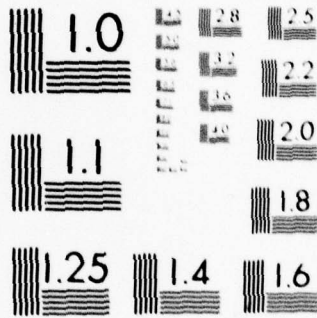
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MEASURE-VALUED PROCESSES IN THE  
CONTROL OF PARTIALLY-OBSERVABLE STOCHASTIC SYSTEMS\*

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# MEASURE-VALUED PROCESSES IN THE CONTROL OF PARTIALLY-OBSERVABLE STOCHASTIC SYSTEMS

WENDELL H. FLEMING

## ABSTRACT

This paper is concerned with the optimal control of continuous-time Markov processes. The admissible control laws are based on white-noise corrupted observations of a function on the state processes. A "separated" control problem is introduced, whose states are probability measures on the original state space. The original and separated control problems are related via the nonlinear filter equation. The existence of a minimum for the separated problem is established. Under more restrictive assumptions it is shown that the minimum expected cost for the separated problem equals the infimum of expected costs for the original problem with partially observed states.

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# MEASURE-VALUED PROCESSES IN THE CONTROL OF PARTIALLY-OBSERVABLE STOCHASTIC SYSTEMS

Wendell H. Fleming

## 1. Introduction.

We are concerned with optimal control of partially-observable stochastic systems, of the following kind. The state (or signal) process is denoted by  $x_t$ ,  $0 \leq t \leq T$ , with  $x_t \in \Sigma$  where  $\Sigma$  is some given "state space". The control process is denoted by  $u_t$ ,  $0 \leq t \leq T$ , with  $u_t \in U$  where  $U$  is some given "control" space. The control  $u_t$  is allowed to depend on observations  $y_s$  for  $0 \leq s \leq t$ . In this paper, we shall assume that

$$(1.1) \quad y_t = \int_0^t h(x_s) ds + w_t,$$

where  $w_t$  is a brownian motion process of some dimension  $v$ . The object is to minimize a criterion of the form  $E\phi(x_T)$ , given an initial distribution of the random variable  $x_0$ .

A precise formulation of the partially-observable control problem is given in §2. An open problem, apparently difficult, is to prove the existence of an optimal control process in case of partial observations. We do not solve this problem here. Instead, we introduce a related control problem in §3, which we call the "separated" problem. In the separated problem the "state" at time  $t$  is a probability measure  $\pi_t$  on  $\Sigma$ . The state process is governed by a stochastic partial differential equation, driven by some  $v$ -dimensional brownian motion  $b_t$  (see (3.1) for this

equation written in a weak form). In the separated problem, the controller is allowed (roughly speaking) complete past observations in choosing the control  $u_t$ . See §3 for the precise formulation. The objective is to minimize  $E\pi_T(\phi)$ , given  $\pi_0$ , where

$$(1.2) \quad \pi(g) = \int_{\Sigma} g(x) d\pi(x).$$

The original control problem with partial observations and the separated problem are related through the nonlinear filter equation (2.5), which is the same as equation (3.1) if  $\pi_t$  is the conditional distribution of  $x_t$  given past observations and  $b_t = \hat{w}_t$  is the innovation.

In §4 we establish some tightness and closure properties associated with the separated problem. Then we prove a result about the existence of a minimum for the separated problem (Theorem 1). The method is an adaptation of [4]. If we let  $\alpha_s$  denote the minimum of  $E\pi_T(\phi)$  in the separated problem and  $\alpha$  be the infimum of  $E\phi(x_T)$  in the original problem, then the nonlinear filter equation implies that  $\alpha_s \leq \alpha$ . In §9 we show that  $\alpha_s = \alpha$ , under fairly restrictive assumptions (Theorem 3). A result like Theorem 3 was proved by Bismut [2] when  $\Sigma$  is a finite set, under still more restrictive conditions.

A separated control problem with state space  $\Sigma$  was also considered by Segall [10]. He considered both observations of the type (1.2) and point observations. A nonlinear semigroup



approach, when  $\Sigma$  is finite, was taken by Davis [3].

Another case of considerable interest is when the state process  $x_t$  obeys a stochastic differential equation

$$(1.3) \quad dx_t = f(x_t, u_t)dt + d\tilde{w}_t,$$

where  $\tilde{w}_t$  is a brownian motion independent of the brownian motion  $w_t$  in (1.2). Unfortunately, our results do not include this case. A minor difficulty is that the state space  $\Sigma$  is some euclidean space, which is not compact as assumed in §2. A more significant difficulty is that the generator  $\mathcal{L}^u$  associated with (1.3) when  $u$  is a constant control is an unbounded operator. The method used to prove Theorem 3 would have to be changed to deal with this case. It is hoped that the device of introducing the separated problem may eventually be useful to study existence of optimal controls for the partially observed control problem.

## 2. The Control Problem with Partial Observations.

Throughout the paper we assume that  $x_t \in \Sigma$ , where  $\Sigma$  is a compact metric space; moreover,  $u_t \in U$ , where  $U$  is a compact, convex subset of euclidean  $R^u$  for some  $u$ . In (1.1) we assume that  $h \in C(\Sigma; R^v)$ ; moreover, in the criterion to be minimized  $\phi \in C(\Sigma)$ , where  $C(\Sigma) = C(\Sigma; R^1)$  is the space of continuous real-valued functions on  $\Sigma$ .

We assume that for each constant control  $u \in U$  there is a semigroup  $\mathcal{T}_t^u$  on  $C(\Sigma)$  associated with a Markov, Feller process  $x_t$ . Let  $\mathcal{L}^u$  be the generator of the semigroup  $\mathcal{T}_t^u$ . We assume that:

(2.1) There is a dense set  $\mathcal{D} \subset C(\Sigma)$  such that  $\mathcal{D} \subset \mathcal{D}(\mathcal{L}^u)$  for all  $u \in U$ . Given  $g \in \mathcal{D}$ ,  $\mathcal{L}^u g \in C(U; C(\Sigma))$ .

(2.2) Given  $g \in C(\Sigma)$  and  $t \geq 0$ ,  $\mathcal{F}_t^u g(x) \in C(\Sigma \times U)$ .

Let  $D([0, T]; \Sigma)$  denote the space of  $\Sigma$ -valued functions which are right continuous and have left hand limits for each  $t \in [0, T]$ ; see [1].

Admissible systems (P.O.). Let  $x, u, w$  be processes defined on some probability space  $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}, P)$ , provided with an increasing family  $\{\mathcal{F}_t\}$  of  $\sigma$ -algebras,  $0 \leq t \leq T$ . For brevity we write  $x$  for the process instead of  $x_t$ ,  $0 \leq t \leq T$ , etc. We require that  $x_t$  is  $\mathcal{F}_t$ -measurable, and that  $w$  is a brownian motion adapted to  $\{\mathcal{F}_t\}$ . Moreover, the paths  $x.(\omega)$  are in  $D([0, T]; \Sigma)$  for each  $\omega \in \Omega$ . For  $g \in \mathcal{D}$  let

$$(2.3) \quad m_t^g = g(x_t) - g(x_0) - \int_0^t \mathcal{L}^u g(x_s) ds.$$

Let  $\mathcal{F}_t^y$  be the  $\sigma$ -algebra generated by  $y_s$ ,  $0 \leq s \leq t$ .

Definition. We say that  $(x, u, w)$  is an admissible system (P.O.) if:

- (i)  $u_t$  is  $\mathcal{F}_t^y$  measurable, for  $0 \leq t \leq T$ .
- (ii) For each  $g \in \mathcal{D}$ ,  $m_t^g$  is a  $\{\mathcal{F}_t\}$ -martingale and  $\langle m^g, w \rangle_t = 0$ .



We recall that the condition  $\langle m^g, w \rangle_t = 0$  is equivalent to requiring that  $m_t^g w_t$  is an  $\{\mathcal{F}_t\}$ -martingale [8]. The partially observed control problem is to find an admissible system  $(x, u, w)$  minimizing  $E\phi(x_T)$ , given  $\phi$  and the distribution of the initial state  $x_0$ .

Since  $x$  has right continuous paths and  $w$  has continuous paths,  $m_t^g$  and  $m_t^g w_t$  are also martingales with respect to  $\{\mathcal{F}_t\}$ . Hence, we may assume that  $\{\mathcal{F}_t\}$  is a right continuous family. In addition, we may complete these  $\sigma$ -algebras by adjoining  $P$ -null subsets of  $\mathcal{F}$ .

In the special case of a constant control  $u$ , we can let  $x$  be a Markov process associated with the semigroup  $\mathcal{F}_t^u$ .

The nonlinear filter equation. Let  $\pi_t$  be a regular conditional distribution for  $x_t$  given  $\mathcal{F}_t^y$ . Given  $g \in C(S)$

$$(2.4) \quad \pi_t(g) = E[g(x_t) | \mathcal{F}_t^y].$$

Since  $\mathcal{F}_0^y$  is the trivial  $\sigma$ -algebra,  $\pi_0$  is the distribution of  $x_0$ . The nonlinear filter equation [7, Theorem 8.1], for  $g \in \mathcal{D}$ , is

$$(2.5) \quad \begin{aligned} \pi_t(g) = & \pi_0(g) + \int_0^t \pi_s(\mathcal{L}^u g) ds \\ & + \int_0^t [\pi_s(gh) - \pi_s(g)\pi_s(h)] \cdot d\hat{w}_s, \end{aligned}$$

where

$$(2.6) \quad \hat{w}_t = y_t - \int_0^t h(x_s) ds$$

is the innovation process. Note that  $gh$  and  $h$  have values in  $R^v$ , and  $\hat{w}_t$  is a  $v$ -dimensional brownian motion adapted to  $\{\mathcal{F}_t^y\}$ . From (2.4) with  $g = \phi$  we have

$$(2.7) \quad E\phi(x_T) = E\{E\phi(x_T) | \mathcal{F}_T^y\} = E\pi_T(\phi).$$

### 3. The Separated Control Problem.

Let  $\mathcal{M} = \mathcal{M}(\Sigma)$  be the space of probability measures on the compact metric space  $\Sigma$ . We give  $\mathcal{M}$  the  $w^*$ -topology; then  $\mathcal{M}$  is compact, metrizable. In the separated problem the "state" process  $\pi_t$  is measure-valued, with  $\pi_t \in \mathcal{M}$ . We define admissible systems for the separated problem as follows. Let  $\pi, u, b$  be processes defined on some probability space  $(\tilde{\Omega}, \mathcal{G}, \{\mathcal{G}_t\}, \tilde{P})$ , provided with an increasing family  $\{\mathcal{G}_t\}$  of  $\sigma$ -algebras,  $0 \leq t \leq T$ . We require that  $\pi_t$  and  $u_t$  are  $\mathcal{G}_t$ -measurable, and that  $b_t$  is a brownian motion of dimension  $v$  adapted to  $\{\mathcal{G}_t\}$

Definition. We say that  $(\pi, u, b)$  is an admissible system (S) if, for each  $g \in \mathcal{D}$ ,

$$(3.1) \quad \pi_t(g) = \pi_0(g) + \int_0^t \pi_s(\mathcal{L}^{u_s} g) ds + \int_0^t [\pi_s(gh) - \pi_s(g)\pi_s(h)] \cdot db_s.$$

The separated control problem is as follows. Given  $\phi \in C(\Sigma)$  and  $\pi_0 \in \mathcal{M}$ , find an admissible system  $(\pi, u, b)$  minimizing  $E\pi_T(\phi)$ .

We emphasize that the separated problem is defined without reference to the partially observed problem in §2. However, equations (2.5)-(2.7) imply the following relationship between the partially observed and separated problems. Given  $\pi_0 \in \mathcal{M}$ , let

$$(3.2) \quad \alpha = \inf\{E\Phi(x_T): (x,u,w) \text{ admissible (P.O.)}, x_0 \text{ has distribution } \pi_0\}$$

$$(3.3) \quad \alpha_s = \inf\{E\pi_T(\Phi): (\pi,u,b) \text{ admissible (S)}, \pi_0 \text{ given}\}$$

If  $(x,u,w)$  is admissible (P.O.), let  $\pi_t$  be a regular conditional distribution of  $x_t$  given  $\mathcal{F}_t^y$ ; let  $\mathcal{G}_t = \mathcal{F}_t^y$ ,  $(\tilde{\Omega}, \mathcal{G}, \tilde{P}) = (\Omega, \mathcal{F}, P)$ . Then  $(\pi, u, \hat{w})$  is admissible (S). By (2.7) and definition of  $\alpha_s$ ,

$$\alpha_s \leq E\pi_T(\Phi) = E\Phi(x_T).$$

Since this is true for all systems admissible (P.O.)

$$(3.4) \quad \alpha_s \leq \alpha.$$

In §9, we will show that  $\alpha_s = \alpha$  under the restrictive assumptions that the generators  $\mathcal{L}^u$  are bounded operators, and that the control  $u$  enters  $\mathcal{L}^u$  linearly.

#### 4. Tightness; Closure Properties (Separated Problem).

If  $(\pi, u, b)$  is an admissible system (S), then by (3.1)  $\pi_t(g)$  is continuous on  $[0, T]$  for each fixed  $g \in \mathcal{D}$ . The same is true for  $g \in C(\Sigma)$ , since  $\mathcal{D}$  is dense in  $C(S)$  and  $\pi_t(\Sigma) = 1$ .



Since  $\mathcal{M}$  has the  $w^*$ -topology, the measure-valued process  $\mathcal{M}$  has paths in  $C([0, T]; \mathcal{M})$ .

Consider any collection  $\mathcal{L}$  of admissible systems  $(\pi, u, b)$ . Let us show tightness of the corresponding collection of probability distributions of  $(\pi, b)$ , which are measures on  $C([0, T]; \mathcal{M}) \times C([0, T]; R^V)$ . This is Lemma 2 below. Let us write  $\pi_*(g)$  for the sample path  $\pi_t(g)$ ,  $0 \leq t \leq T$ .

Lemma 1. For every  $g \in \mathcal{D}$ ,  $\epsilon > 0$  there exists a compact set  $B_{\epsilon g} \subset C([0, T])$  such that

$$P(\pi_*(g) \in B_{\epsilon g}) \geq 1 - \epsilon.$$

Proof. By (3.1)

$$\pi_t(g) - \pi_r(g) = F_g(t) - F_g(r) + M_g(t) - M_g(r),$$

$$F_g(t) = \int_0^t \pi_s(\mathcal{L}^u_s g) ds$$

$$M_g(t) = \int_0^t [\pi_s(gh) - \pi_s(g)\pi_s(h)] \cdot db_s.$$

We have  $|\pi_s(\mathcal{L}^u_s g)| \leq \|\mathcal{L}^u_s g\| \leq K_g$ , by assumption (2.1) and compactness of the control space  $U$ . Hence,  $F_g(\cdot)$  is Lipschitz with constant  $K_g$ , and  $F_g(0) = 0$ . Moreover,  $M_g(t)$  is a martingale with increasing process

$$\langle M_g(t) \rangle = \int_0^t |\pi_s(gh) - \pi_s(g)\pi_s(h)|^2 ds.$$

Since  $|\pi_s(gh) - \pi_s(g)\pi_s(h)|^2 \leq K_{1g}$ , we have  $\langle M_g(\cdot) \rangle$  Lipschitz with constant  $K_{1g}$ . From these facts, Lemma 1 follows by well-known arguments [9, Proposition 9] [5, Lemma 4].

Lemma 2. For every  $\epsilon > 0$  there exist compact sets  $A_{\epsilon 1} \subset C([0, T]; \mathcal{M})$  and  $A_{\epsilon 2} \subset C([0, T]; R^V)$  such that

$$P(\pi. \in A_{\epsilon 1}) \geq 1 - \epsilon, \quad P(b. \in A_{\epsilon 2}) \geq 1 - \epsilon.$$

The existence of  $A_{\epsilon 1}$  follows from Lemma 1 and elementary properties of the  $w^*$ -topology on  $\mathcal{M}$ ; see [5, Lemma 3]. The existence of  $A_{\epsilon 2}$  is a known property of brownian motion.

Closure results. Let us consider sequences of admissible systems  $(\pi_n, u_n, b_n)$ ,  $n = 1, 2, \dots$ , all defined on the same  $(\tilde{\Omega}, \mathcal{G}, \{\mathcal{G}_t\}, \tilde{P})$ . If this sequence has a limit  $(\pi, u, b)$ , in a suitable sense, we wish to give conditions under which  $(\pi, u, b)$  is admissible. In the first closure result we consider constant controls  $u_{nt} \equiv u_n$ .

We recall that  $\mathcal{M}$  with the  $w^*$ -topology is metrizable. Hence, one can consider uniform convergence on  $[0, T]$  of sequences  $\pi_{nt}$ . This is equivalent to uniform convergence of  $\pi_{nt}(g)$  to  $\pi_t(g)$  for each  $g \in C(\Sigma)$ .

Lemma 3. Let  $(\pi_n, u_n, b_n)$  be admissible (S),  $n = 1, 2, \dots$ , with  $u_n \in U$  a constant control such that  $u_n \rightarrow u$  as  $n \rightarrow \infty$ . Suppose that  $\pi_{nt} \rightarrow \pi_t$ ,  $b_{nt} \rightarrow b_t$  uniformly on  $[0, T]$  as  $n \rightarrow \infty$ ,



with probability 1. Then  $(\pi, u, b)$  is admissible (S).

Proof. For each  $g \in \mathcal{D}$

$$\pi_{nt}(g) = \pi_{n0}(g) + \int_0^t \pi_{ns}(\mathcal{L}^{u_n}_g) ds + \int_0^t e_{ns} \cdot db_{ns},$$

$$e_{ns} = \pi_{ns}(gh) - \pi_{ns}(g)\pi_{ns}(h).$$

We have since  $\pi_{ns}(\Sigma) = 1$

$$|\pi_{ns}(\mathcal{L}^{u_n}_g) - \pi_s(\mathcal{L}^u_g)| \leq ||\mathcal{L}^{u_n}_g - \mathcal{L}^u_g|| + |\pi_{ns}(\mathcal{L}^u_g) - \pi_s(\mathcal{L}^u_g)|.$$

By (2.1),  $||\mathcal{L}^{u_n}_g - \mathcal{L}^u_g|| \rightarrow 0$ ; and since  $\mathcal{L}^u_g \in C(\Sigma)$ ,  $\pi_{ns}(\mathcal{L}^u_g) \rightarrow \pi_s(\mathcal{L}^u_g)$  uniformly on  $[0, T]$  with probability 1. Hence, with probability 1

$$\lim_{n \rightarrow \infty} \int_0^t \pi_{ns}(\mathcal{L}^{u_n}_g) ds = \int_0^t \pi_s(\mathcal{L}^u_g) ds, \quad 0 \leq t \leq T.$$

Moreover,  $e_{ns}$  is uniformly bounded and tends uniformly on  $[0, T]$  to  $e_s = \pi_s(gh) - \pi_s(g)\pi_s(h)$ , with probability 1.

By Lemma 1, given  $\varepsilon > 0$  there exists a compact set  $D_\varepsilon \subset C([0, T]; \mathbb{R}^V)$  such that  $\tilde{P}(e_n \in D_\varepsilon) \geq 1 - \varepsilon$ . Since compact subsets of  $C([0, T]; \mathbb{R}^V)$  are equicontinuous, it follows by using piecewise constant approximations that as  $n \rightarrow \infty$

$$\int_0^t e_{ns} \cdot db_{ns} \rightarrow \int_0^t e_s \cdot db_s \quad \text{in probability.}$$

See [4, pp. 789-790]. Therefore,  $(\pi, u, b)$  satisfies (3.1) for each  $g \in \mathcal{D}$ , which shows that  $(\pi, u, b)$  is admissible. This proves Lemma 3.

Let us now establish a second closure result, which will be used in §5 for the proof of an existence theorem for an optimal control in the separated problem. We recall that  $u = (u_1, \dots, u_\mu) \in U$ , where  $U \subset \mathbb{R}^\mu$ . We now impose the following assumption on the form of the generators  $\mathcal{L}^u$ :

$$(4.1) \quad \mathcal{L} = \mathcal{L}^0 + \mathcal{L}^1 \cdot u, \text{ where } \mathcal{L}^0: \mathcal{D}(\mathcal{L}^0) \rightarrow C(\Sigma), \\ \mathcal{L}^1: \mathcal{D}(\mathcal{L}^1) \rightarrow C(\Sigma; \mathbb{R}^\mu) \text{ are linear operators with } \mathcal{D} \subset \mathcal{D}(\mathcal{L}^i), \\ i = 0, 1.$$

Note that (4.1) implies (2.1).

When (4.1) holds, let

$$v_t = \int_0^t u_s ds.$$

If  $(\pi, u, b)$  is an admissible system (S), let us call  $(\pi, v, b)$  an admissible system (S'). Equation (3.1) can now be rewritten as

$$(4.2) \quad \pi_t(g) = \pi_0(g) + \int_0^t \pi_s(\mathcal{L}^0 g) ds + \int_0^t \pi_s(\mathcal{L}^1 g) \cdot dv_s \\ + \int_0^t [\pi_s(gh) - \pi_s(g)\pi_s(h)] \cdot db_s.$$

Thus, the conditions that  $(\pi, v, b)$  be admissible (S') are that

$\pi_t, v_t$  be  $\mathcal{G}_t$ -measurable, that  $b_t$  be a brownian motion adapted to  $\{\mathcal{G}_t\}$ , that (4.2) hold for every  $g \in \mathcal{D}$ , and that  $u_t = dv_t/dt$  is in  $U$  almost everywhere on  $[0, T]$  with probability 1.

Since  $U$  is compact,  $|u_s| \leq N$  for some  $N$ . Hence,  $v_\cdot$  is Lipschitz with constant  $N$ . Since  $v_0 = 0$ ,  $v_\cdot$  lies in a fixed compact subset of  $C([0, T]; \mathbb{R}^\mu)$ . We then have by Lemma 2:

Lemma 4. For every  $\epsilon > 0$  there exists a compact set  $A_\epsilon \subset C([0, T]; \mathcal{M} \times \mathbb{R}^\mu \times \mathbb{R}^\nu)$  such that  $P((\pi_\cdot, v_\cdot, b_\cdot) \in A_\epsilon) \geq 1 - \epsilon$ .

The second closure result is:

Lemma 5. Let  $(\pi_n, v_n, b_n)$  be admissible  $(S')$ ,  $n = 1, 2, \dots$ . Suppose that (4.1) holds and that  $(\pi_{nt}, v_{nt}, b_{nt}) \rightarrow (\pi_t, v_t, b_t)$  uniformly on  $[0, T]$  as  $n \rightarrow \infty$ , with probability 1. Then  $(\pi, v, b)$  is admissible  $(S')$ .

Proof. Consider any  $g \in \mathcal{D}$ . Since  $\mathcal{L}^0 g \in C(S)$ ,  $\pi_{ns}(\mathcal{L}^0 g) \rightarrow \pi_s(\mathcal{L}^0 g)$  uniformly on  $[0, T]$ . Similarly  $\pi_{ns}(\mathcal{L}^1 g) \rightarrow \pi_s(\mathcal{L}^1 g)$  uniformly on  $[0, T]$ . Since  $v_{ns} \rightarrow v_s$  uniformly on  $[0, T]$  and  $|dv_{nt}/dt| \leq N$ , we have

$$\lim_{n \rightarrow \infty} \int_0^t \pi_{ns}(\mathcal{L}^0 g) ds = \int_0^t \pi_s(\mathcal{L}^0 g) ds$$

$$\lim_{n \rightarrow \infty} \int_0^t \pi_{ns}(\mathcal{L}^1 g) \cdot dv_{ns} = \int_0^t \pi_s(\mathcal{L}^1 g) \cdot dv_s.$$

The same proof as Lemma 3 then shows that  $(\pi, v, b)$  satisfies (4.2),



for each  $g \in \mathcal{D}$ . Finally, since  $u_{nt} = dv_n/dt$  is in  $U$ ,  $U$  is compact and convex, and  $v_{nt} \rightarrow v_t$  uniformly on  $[0, T]$ , we have  $dv_t/dt \in U$ . Hence,  $(\pi, v, b)$  is admissible  $(S')$ .

#### 5. An Existence Theorem (Separated Problem).

In order to show that there is a minimum in the separated problem, we show that there is an admissible system  $(S')$  for which the infimum  $\alpha_s$  in (3.3) is attained. This will be proved using results in §4, following the method of [4]. The distribution of a triple  $(\pi, v, b)$  is a probability measure on  $C([0, T]; \mathcal{M} \times R^\mu \times R^\nu)$ . Triples  $(\pi, v, b), (\bar{\pi}, \bar{v}, \bar{b})$  with the same distribution measure are identical in distribution.

Theorem 1. Suppose that (4.1) holds. Then there exists an admissible system  $(S')$   $(\bar{\pi}, \bar{v}, \bar{b})$  such that  $E\bar{\pi}_T(\phi) = \alpha_s$ .

Proof. Let  $(\pi_n, v_n, b_n)$  be a minimizing sequence  $(S')$ ; thus  $E\pi_{nT}(\phi) \geq \alpha_s$  and  $E\pi_{nT}(\phi) \rightarrow \alpha_s$  as  $n \rightarrow \infty$ . By Lemma 4 and Skorokhod's theorem, there exist a subsequence of  $n$  and  $(\bar{\pi}_n, \bar{v}_n, \bar{b}_n)$  identical in distribution with  $(\pi_n, v_n, b_n)$  such that  $\bar{\pi}_{nt}, \bar{v}_{nt}, \bar{b}_{nt}$  tend to limits  $\bar{\pi}_t, \bar{v}_t, \bar{b}_t$  uniformly on  $[0, T]$ , with probability 1. By Lemma 5,  $(\bar{\pi}, \bar{v}, \bar{b})$  is an admissible system  $(S')$ . Moreover

$$\alpha_s = \lim_{n \rightarrow \infty} E\bar{\pi}_{nT}(\phi) = E\bar{\pi}_T(\phi).$$

This proves Theorem 1.

## 6. Constant Controls.

The remainder of this paper is concerned with the relationship between the infima  $\alpha$  and  $\alpha_s$  in (3.2), (3.3). For this purpose, we consider piecewise constant controls in §'s 7 and 8. In preparation, let us suppose in this section that  $u$  is a constant control,  $u_t = u$  for  $0 \leq t \leq T$ .

In §'s 6-8 we do not use linearity of  $\mathcal{L}^u$  in  $u$  in (4.1). Instead, we make the general assumptions (2.1) and (2.2).

Lemma 6. Given  $\pi_0 \in \mathcal{M}$ ,  $u \in U$ , and a brownian motion  $b$ , there exists an  $\mathcal{M}$ -valued process  $\pi$  which is a solution to (3.1) for all  $g \in \mathcal{D}$ . Moreover, the distribution of  $\pi$  is unique (it depends only on  $\pi_0$  and  $u$ ).

Lemma 6 follows from results of Kunita [6] and Szpirglas [11]. Kunita's construction [6, p. 374] gives uniqueness in distribution to the corresponding equation for  $\pi_t$  written in terms of the semigroup  $\mathcal{T}_t^u$  on  $C(S)$  generated by  $\mathcal{L}^u$ . In [11, Th. III.1] Szpirglas showed that that equation is equivalent to (3.1).

Lemma 7. Given  $\pi_0 \in \mathcal{M}$ ,  $u \in U$ , and  $F \in C(\mathcal{M})$ , let  $\psi(\pi_0, u; F, T) = EF(\pi_T)$ . Then  $\psi$  is continuous on  $\mathcal{M} \times U$ .

Proof. Let  $\pi_{n0} \rightarrow \pi_0$ ,  $u_n \rightarrow u$  ( $u_n \in U$ ); and let  $(\pi_n, u_n, b_n)$  be admissible (S) with  $\pi_{n0}$  the state of the process  $\pi_{nt}$  when  $t = 0$ . By Lemma 2 and Skorokhod's theorem, there exist  $(\bar{\pi}_n, \bar{b}_n)$  identical in distribution to  $(\pi_n, b_n)$  and a subsequence of  $n$



such that  $u_n \rightarrow u$ ,  $(\bar{\pi}_{nt}, \bar{b}_{nt}) \rightarrow (\bar{\pi}_t, \bar{b}_t)$  uniformly on  $[0, T]$  with probability 1. By Lemma 3,  $(\bar{\pi}, u, \bar{b})$  is admissible (S). Moreover,  $\bar{\pi}_0 = \pi_0$ . Then

$$\lim_{n \rightarrow \infty} \psi(\pi_{n0}, u_n; F, T) = \lim_{n \rightarrow \infty} EF(\bar{\pi}_{nT}) = EF(\bar{\pi}_T) = \psi(\pi_0, u; F, T).$$

This proves Lemma 7.

Note that in defining  $\psi(\pi_0, u; F, T)$ , we have used the uniqueness in distribution of  $\pi_T$ , which is implied by Lemma 6.

From Lemma 7 and compactness of  $\mathcal{M} \times U$  we have:

Corollary.  $V(\pi) = \min_{u \in U} \psi(\pi, u; F, T)$  is continuous on  $\mathcal{M}$ .

#### 7. $\Delta$ -Admissible Systems (S).

In this section and in §8, we let  $\Delta$  denote a fixed partition of  $[0, T]$  into subintervals  $[t_k, t_{k+1}]$ , with  $0 = t_0 < t_1 < \dots < t_m = T$ . We define  $V_k(\pi)$  by backward induction on  $k$ . For  $\pi \in \mathcal{M}$

$$(7.1) \quad V_m(\pi) = \pi(\phi)$$

$$(7.2) \quad V_k(\pi) = \min_{u \in U} \psi(\pi, u; V_{k+1}, t_{k+1} - t_k), \quad k = 0, 1, \dots, m-1.$$

By the Corollary in §6,  $V_k \in C(\mathcal{M})$ .

Equation (7.2) is a discrete-time dynamic programming equation for the separated control problem, with constant control on each interval  $[t_k, t_{k+1})$ , in a sense which we shall indicate below.

Definition. An admissible system  $(\pi, u, b)$  is  $\Delta$ -admissible (S). if  $u_t$  is constant for  $t_k \leq t < t_{k+1}$ ,  $k = 0, 1, \dots, m-1$ .

We recall that an admissible system (S) is defined on some  $(\tilde{\Omega}, \mathcal{G}, \{\mathcal{G}_t\}, \tilde{P})$ .

Lemma 8. If  $(\pi, u, b)$  is  $\Delta$ -admissible, then

$$V_k(\pi_{t_k}) \leq E\{\pi_T(\phi) | \mathcal{G}_{t_k}\}, \tilde{P} - \text{a.s.}$$

Proof. We use backward induction on  $k$ . For  $k = m$ ,  $t_m = T$ ,

$$V_m(\pi_T) = \pi_T(\phi) = E\{\pi_T(\phi) | \mathcal{G}_T\}, \tilde{P} - \text{a.s.}$$

Suppose Lemma 8 is true for  $k+1$ . Then

$$(*) \quad E\{\pi_T(\phi) | \mathcal{G}_{t_k}\} = E\{E\{\pi_T(\phi) | \mathcal{G}_{t_{k+1}}\} | \mathcal{G}_{t_k}\} \geq E\{V_{k+1}(\pi_{t_{k+1}}) | \mathcal{G}_{t_k}\}.$$

Let  $\zeta$  denote  $(\pi, u, b)$  restricted to  $[t_k, t_{k+1})$ ; and let  $\Gamma_k = \Gamma_k(\omega; \cdot)$  be a regular conditional distribution of this triple given  $\mathcal{G}_{t_k}$ . With  $\tilde{P}$ -probability 1,  $u_t$  is a constant  $u_k$  on  $[t_k, t_{k+1})$  and  $\pi_{t_k}$  is constant  $\Gamma_k$ -almost surely. Moreover, the restriction of  $b_t$  to  $[t_k, t_{k+1}]$  is a  $\Gamma_k$ -brownian motion. Let

$$\begin{aligned} G(\zeta, t) &= \pi_t(g) - \pi_{t_k}(g) - \int_{t_k}^t \pi_s(\mathcal{L}^{u_k} g) ds \\ &\quad - \int_{t_k}^t [\pi_s(gh) - \pi_s(g)\pi_s(h)] \cdot db_s. \end{aligned}$$

Let  $G(\zeta) = \max_{[t_k, t_{k+1}]} |G(\zeta, t)|$ . By (3.1),  $G(\zeta) = 0$ ,  $\tilde{P}$ -almost surely.

Hence,  $G(\zeta) = 0$ ,  $\Gamma_k$ -almost surely, with  $\tilde{P}$ -probability 1. With respect to the measure  $\Gamma_k, \zeta$  is a solution of  $G(\zeta, t) = 0$  on  $[t_k, t_{k+1}]$ , for each  $g \in \mathcal{D}$ . Hence,  $P$ -almost surely

$$\begin{aligned} E\{V_{k+1}(\pi_{t_{k+1}}) | \mathcal{G}_{t_k}\} &= E^{\Gamma_k} V_{k+1}(\pi_{t_{k+1}}) = \psi(\pi_{t_k}, u_k; V_{k+1}, t_{k+1} - t_k) \\ &\geq V_k(\pi_{t_k}). \end{aligned}$$

This, together with (\*), proves Lemma 8.

Since  $\pi_0$  is given (not random) Lemma 8 implies when  $k = 0$

$$(7.3) \quad V_0(\pi_0) \leq E\{\pi_T(\Phi)\}.$$

Let

$$(7.4) \quad \alpha_S^\Delta = \inf\{E\pi_T(\Phi) : (\pi, u, b) \text{ } \Delta\text{-admissible (S), } \pi_0 \text{ given}\}.$$

Then (7.3) implies that  $V_0(\pi_0) \leq \alpha_S^\Delta$ . In fact,  $V_0(\pi_0) = \alpha_S^\Delta$ . This follows from Theorem 2 in the next section. A direct proof that  $V_0(\pi_0) = \alpha_S^\Delta$  could also be given in terms of the separated problem only without reference to admissible systems (P.O.); but we shall not do so.

In a similar way,  $V_k(\pi_k)$  is the infimum of  $E\pi_T(\Phi)$  for a separated problem on  $[t_k, T]$ , using controls constant on intervals  $[t_\ell, t_{\ell+1})$ ,  $\ell \geq k$ , and with  $\pi_{t_k} = \pi_k$ . This justifies calling (7.2) a discrete-time dynamic programming equation.



8.  $\Delta$ -Admissible Systems (P.O.). As in §7, let  $\Delta$  be a fixed partition of  $[0, T]$ .

Definition. An admissible system  $(x, u, w)$  is  $\Delta$ -admissible (P.O.) if  $u_t$  is constant for  $t_k \leq t < t_{k+1}$ ,  $k = 0, 1, \dots, m - 1$ .

Given  $\pi_0 \in \mathcal{M}$  let

$$(8.1) \quad \alpha^\Delta = \inf E\phi(x_T): (x, u, w) \Delta\text{-admissible (P.O.)}, x \text{ has distribution } \pi_0\}.$$

As in (3.4) we have  $\alpha_s^\Delta \leq \alpha^\Delta$ . The purpose of the present section is to prove:

$$\text{Theorem 2. } \alpha^\Delta = \alpha_s^\Delta = V_0(\pi_0).$$

Since  $V_0(\pi_0) \leq \alpha_s^\Delta \leq \alpha^\Delta$ , it is sufficient to prove that, for any  $\varepsilon > 0$ , there exists  $(x, u, w)$   $\Delta$ -admissible (P.O.) such that  $x_0$  has distribution  $\pi_0$  and

$$(8.2) \quad E\{\phi(x_T)\} \leq V_0(\pi_0) + \varepsilon.$$

This follows from Lemma 10 below.

We begin with the following construction, similar to one used by Bensoussan-Lions [12]. Let  $\mathcal{O}_1, \dots, \mathcal{O}_\ell$  be disjoint, Borel measurable subsets of  $\mathcal{M}$ , with  $\mathcal{M} = \mathcal{O}_1 \cup \dots \cup \mathcal{O}_\ell$ . Let  $u_{kj} \in U$ ,  $k = 0, 1, \dots, m - 1$ ,  $j = 1, \dots, \ell$ . Given an initial distribution  $\pi_0$  for  $x_0$ , we wish to construct a  $\Delta$ -admissible system (P.O.)  $(x, u, w)$  with the property

$$(8.3) \quad u_t = u_{kj} \quad \text{if} \quad \pi_{t_k} \in \mathcal{O}_j, \quad t_k \leq t < t_{k+1}, \quad k = 0, 1, \dots, m-1,$$

where  $\pi_t$  is a regular conditional distribution of  $x_t$  given  $\mathcal{F}_t^y$ .

The system  $(x, u, w)$  will be defined on the "canonical" sample space

$$\Omega = D([0, T]; \Sigma) \times C([0, T]; R^V),$$

whose elements we denote by  $x, w$ . Let  $\mathcal{F}_t$  be generated by  $x, w$  paths for  $0 \leq s \leq t$ , with  $\mathcal{F} = \mathcal{F}_T$ . We define by induction a sequence of probability measures  $P_0, P_1, \dots, P_{m-1}$  as follows; then we take  $P = P_{m-1}$ . The measure  $P_k$  will be defined on  $\mathcal{F}_{t_{k+1}}$ .

Let  $Q_{xk}^u$  be the probability distribution on  $D([t_k, t_{k+1}]; \Sigma)$  of a Markov process with initial state  $x_{t_k} = x$  and generator  $\mathcal{L}^u$ . From assumption (2.2) and the Markov property  $Q_{xk}^u$  depends continuously on  $(x, u) \in \Sigma \times U$  in the sense of convergence of finite dimensional distributions. Let  $W_{wk}$  be Wiener measure on  $C([t_k, t_{k+1}]; R^V)$  for paths starting at  $w_{t_k} = w$ .

For  $0 \leq t \leq t_1$ , the control is constant:  $u_t = u_0 = u_{0j}$  for that  $j$  such that  $\pi_0 \in \mathcal{O}_j$ . We define  $P_0$  on  $\mathcal{F}_{t_1}$  as the product measure  $P_0 = Q_0 \times W_{00}$ , where

$$Q_0(B) = \int_{\Sigma} Q_{x0}^{u_0}(B) d\pi_0(x), \quad B \in \mathcal{F}_{t_1}.$$

Now suppose that  $P_0, P_1, \dots, P_{k-1}$  have been defined, as well as piecewise constant controls  $u_t$  for  $0 \leq t < t_k$ . As in (8.3),



we define  $u_t = u_{kj}$  if  $\pi_{t_k} \in \mathcal{O}_j$ ,  $t_k \leq t < t_{k+1}$ , where  $\pi_{t_k}$  is a regular conditional distribution of  $x_{t_k}$  given  $\mathcal{F}_{t_k}^y$  with respect to the measure  $P_{k-1}$ . Let  $(x', w')$ ,  $(x'', w'')$  denote the restrictions to  $[0, t_k]$  and  $(t_k, t_{k+1}]$  respectively of  $(x, w)$ . The measure  $P_k$  is defined first for subsets of  $\Omega$  the form  $B' \times B''$ , where  $B' \in \mathcal{F}_{t_k}$  is generated by  $(x', w')$  paths and  $B''$  is a "window set" of the form  $B'' = \{(x'', w'') : (x_{s_i}, w_{s_i}) \in A_i\}$  for finitely many  $s_i \in (t_k, t_{k+1}]$ . Let  $Q_k = Q_{x_{t_k}}$  and  $W_k = W_{w_{t_k}}$  where  $u_k = u_t$  for  $t_k \leq t < t_{k+1}$ . Then

$$(8.4) \quad P_k(B' \times B'') = \int_{B'} (Q_k \times W_k)(B'') dP_{k-1}(x', w').$$

This determines the probability measure  $P_k$  on  $\mathcal{F}_{t_{k+1}}$ .

We take  $P = P_{m-1}$ .

Lemma 9. The system  $(x, u, w)$  is  $\Delta$ -admissible (P.O.).

Proof. By construction,  $u_t$  is  $\mathcal{F}_t^y$ -measurable and constant on each interval of the partition  $\Delta$ . According to the definition in §2 it suffices to verify that, for each  $g \in \mathcal{D}$ ,  $m_t^g$  and  $m_t^g w_t$  are  $\{\mathcal{F}_t\}$ -martingales, where  $m_t^g$  is defined by (2.3). Let us first consider  $t_k \leq r < t < t_{k+1}$ . Let  $\mathcal{F}_r''$  be the  $\sigma$ -algebra generated by  $x'', w''$  paths restricted to  $(t_k, r]$ . Then

$$\begin{aligned} E^{Q_k \times W_k} \{ (m_t^g - m_r^g) | \mathcal{F}_r'' \} \\ = E^{Q_k} \{ g(x_t) - g(x_r) - \int_r^t \mathcal{L}^{u_k} g(x_s) ds | \mathcal{F}_r'' \} = 0. \end{aligned}$$

Hence, for any  $B'' \in \mathcal{F}_r''$ ,

$$\int_{B''} (m_t^g - m_r^g) d(Q_k \times W_k) = 0.$$

Since  $m_t^g - m_r^g$  and  $w_t - w_r$  are independent with respect to  $Q_k \times W_k$ , we also have

$$\int_{B''} (m_t^g - m_r^g) (w_t - w_r) d(Q_k \times W_k) = 0.$$

For  $B \in \mathcal{F}_r$  of the form  $B = B' \times B''$  with  $B' \in \mathcal{F}_{t_k}'$  and  $B'' \in \mathcal{F}_r''$

$$\int_B (m_t^g - m_r^g) dP_k = \int_{B'} \int_{B''} (m_t^g - m_r^g) d(Q_k \times W_k) dP_{k-1} = 0.$$

Thus,  $E\{m_t^g | \mathcal{F}_r\} = m_r^g$ . Similarly,

$$E\{(m_t^g - m_r^g)(w_t - w_r) | \mathcal{F}_r\} = 0,$$

from which  $E\{m_t^g w_t | \mathcal{F}_r\} = m_r^g w_r$ , for  $t_k \leq r < t < t_{k+1}$ . If  $r < t_k \leq t \leq t_{k+1}$ , we first condition on  $\mathcal{F}_{t_k}$  and then on  $\mathcal{F}_r$ . This shows that  $m_t^g$  and  $m_t^g w_t$  are  $\{\mathcal{F}_t\}$ -martingales, from which Lemma 9 follows.

Lemma 10. Given  $\epsilon > 0$  there exists an admissible system (P.O.)  
(x,u,w) such that

$$E\{\pi_T(\phi) | \mathcal{F}_{t_k}^y\} \leq V_k(\pi_{t_k}) + \epsilon(1 - \frac{k}{m})$$

for  $k = 0, 1, \dots, m-1$  and with any distribution  $\pi_0$  for  $x_0$ . Here  $\pi_{t_k}$  is a regular conditional distribution for  $x_{t_k}$  given  $\mathcal{F}_{t_k}^y$ .

Proof. In the construction above we choose the partition  $\mathcal{O}_1, \dots, \mathcal{O}_\ell$  of  $\mathcal{M}$  fine enough and  $u_{kj}$  such that

$$\psi(\pi, u_{kj}; V_{k+1}, t_{k+1} - t_k) \leq V_k(\pi) + \frac{\varepsilon}{m}$$

for all  $\pi \in \mathcal{O}_j$ ,  $k = 0, 1, \dots, m$ . This is possible by Lemma 7 and compactness of  $\mathcal{M} \times U$ . The probability measure  $P = P_{m-1}$  and the piecewise-constant control process  $u_t$  on the canonical sample space are defined by the construction. We now proceed by backward induction on  $k$  (compare with the proof of Lemma 8). For  $k = m$ ,  $V_m(\pi_T) = \pi_T(\Phi)$ . Proceeding inductively, suppose Lemma 10 true for  $k + 1$ . Let  $\mathcal{O}_k$  be a regular conditional distribution for  $(x, u, w)$  given  $\mathcal{F}_{t_k}^y$ . With  $P$ -probability 1,  $u_t$  is a constant  $u_{kj}$  on  $[t_k, t_{k+1})$  and  $\pi_{t_k}$  is constant  $\mathcal{O}_k$ -almost surely. Moreover, by (2.5)

$$\pi_t(g) = \pi_{t_k}(g) + \int_{t_k}^t \pi_s(\mathcal{L}^{u_{kj}} g) ds + \int_{t_k}^t [\pi_s(gh) - \pi_s(g)\pi_s(h)] \cdot d\hat{w}_s,$$

for  $t_k \leq t \leq t_{k+1}$ . Hence, by the definition of  $\psi$  in Lemma 7,

$$E^{\mathcal{O}_k} V_{k+1}(\pi_{t_{k+1}}) = \psi(\pi_{t_k}, u_{kj}; V_{k+1}, t_{k+1} - t_k).$$

We then have, with  $P$ -probability 1,

$$E(V_{k+1}(\pi_{t_{k+1}}) | \mathcal{F}_{t_k}) = E^{\mathcal{O}_k} V_{k+1}(\pi_{t_{k+1}}) \leq V_k(\pi_{t_k}) + \frac{\varepsilon}{m},$$



$$\begin{aligned}
E\{\pi_T(\phi) | \mathcal{F}_{t_k}\} &= E\{E\{\pi_T(\phi) | \mathcal{F}_{t_{k+1}}\} | \mathcal{F}_{t_k}\} \\
&\leq E\{V_{k+1}(\pi_{t_{k+1}}) | \mathcal{F}_{t_k}\} + \epsilon(1 - \frac{k-1}{m}) \\
&\leq V_k(\pi_{t_k}) + \epsilon(1 - \frac{k}{m}).
\end{aligned}$$

This proves Lemma 10.

We get the inequality (8.2) needed to prove Theorem 2 by taking  $k = 0$  in Lemma 10, since  $\mathcal{F}_0^Y$  is the trivial  $\sigma$ -algebra and  $E\pi_T(\phi) = E\phi(x_T)$ .

#### 9. A Sufficient Condition for $\alpha = \alpha_s$ .

According to (3.4),  $\alpha_s \leq \alpha$ ; while by Theorem 2,  $\alpha_s^\Delta = \alpha^\Delta$ . Since the class of  $\Delta$ -admissible systems is contained in the class of admissible systems (either (P.O.) or (S)), we also have  $\alpha^\Delta \geq \alpha$  and  $\alpha_s^\Delta \geq \alpha_s$ . Therefore, we will have  $\alpha_s = \alpha$  if we can show that

$$(9.1) \quad \alpha_s = \inf_{\Delta} \alpha_s^\Delta.$$

Unfortunately, we have verified (9.1) only under the rather restrictive assumptions of Theorem 3 below.

The proof of Theorem 3 will proceed as follows. Given any admissible system (S)  $(\pi, u, b)$ , approximations  $(\pi^n, u^n, b)$  are made such that  $u_t^n$  is piecewise constant. It is shown that  $\pi_t^n$  is near  $\pi_t$ , in a suitable sense, if  $E \int_0^T |u_t^n - u_t|^2 dt$  is small. See (9.5). However, our proof of this estimate uses boundedness of the generators  $\mathcal{L}^u$ . To simplify matters we also assume that the control enters

linearly,  $\mathcal{L}^u = \mathcal{L}^0 + \mathcal{L}^1 \cdot u$  as in (4.1).

Theorem 3. Assume that (4.1) holds and that  $\mathcal{L}^0, \mathcal{L}^1$  are bounded operators on  $C(S)$ . Then  $\alpha = \alpha_s$ .

Proof. Let  $(\pi, u, b), (\pi^n, u^n, b)$  be admissible (S) with  $\pi_0^n = \pi_0$ . Let  $\phi_t = \pi_t - \pi_t^n$ ,  $\theta_t(g) = E[\phi_t(g)]^2$ ; both  $\phi_t, \theta_t$  depend of course on  $n$ . From (4.2) we have

$$\begin{aligned}
 \phi_t(g) &= \int_0^t \phi_s(\mathcal{L}^0 g) ds + \int_0^t \phi_s(\mathcal{L}^1 g) \cdot u_s ds \\
 &\quad + \int_0^t \pi_s^n(\mathcal{L}^1 g) \cdot (u_s - u_s^n) ds \\
 &\quad + \int_0^t [\phi_s(gh) - \phi_s(g)\pi_s(h) - \phi_s(h)\pi_s^n(g)] \cdot db_s, \\
 (9.2) \quad \theta_t(g) &\leq K \left\{ \int_0^t [\theta_s(\mathcal{L}^0 g) + \theta_s(\mathcal{L}^1 g)] ds \right. \\
 &\quad + ||\mathcal{L}^1 g||^2 E \int_0^t |u_s - u_s^n|^2 ds \\
 &\quad \left. + \int_0^t [\theta_s(gh) + ||h||^2 \theta_s(g) + ||g||^2 \theta_s(h)] ds \right\},
 \end{aligned}$$

for some constant  $K$ .

Let  $h = g_0$ ; take  $g_1, g_2, \dots$  such that  $||g_j|| \leq C$  and their linear combinations are dense in  $C(S)$ . Let  $\mathcal{L}^2 g = hg$ ,

$$\mathcal{J}_{0j} = \{g_j\}, \quad \mathcal{J}_{1j} = \{\mathcal{L}^i g_j, \quad i = 0, 1, 2\},$$

$$\dots, \mathcal{J}_{kj} = \{\mathcal{L}^i f, \quad i = 0, 1, 2: f \in \mathcal{J}_{k-1,j}\},$$

where  $j = 0, 1, 2, \dots$ . Let

$$\beta_{kj}(t) = \max_{f \in \mathcal{J}_{kj}} \theta_t(f)$$

$$\gamma_{kj} = \max_{f \in \mathcal{J}_{kj}} \|f\|^2$$

$$\|u - u^n\|_2^2 = E \int_0^T |u_t - u_t^n|^2 dt.$$

Since  $\mathcal{L}^0, \mathcal{L}^1$  are bounded operators and  $\|h\| < \infty$ , we have for some  $M$

$$\gamma_{k+1,j} \leq M\gamma_{kj}, \quad \gamma_{0j} \leq C^2.$$

Hence,  $\gamma_{kj} \leq C^2 M^k$ . In (9.2) we take  $g \in \mathcal{J}_{kj}$ . Since  $\mathcal{L}^1$  is a bounded operator we have for some  $K_1$

$$(9.3) \quad \beta_{kj}(t) \leq K_1 \left[ \int_0^t [\beta_{k+1,j}(s) + \beta_{kj}(s) + \gamma_{kj} \beta_{00}(s)] ds + \gamma_{kj} \|u - u^n\|_2^2 \right].$$

Take  $\rho > M$ , and let

$$(9.4) \quad \beta_j(t) = \sum_{k=0}^{\infty} \rho^{-k} \beta_{kj}(t),$$

$$\beta(t) = \sum_{j=0}^{\infty} 2^{-j} \beta_j(t).$$

From (9.3)



$$\begin{aligned} \beta_j(t) \leq K_1 \left[ \int_0^t (\rho+1) \beta_j(s) ds + \int_0^t \beta_{00}(s) \left( \sum_{k=0}^{\infty} \rho^{-k} C^2 M^k \right) ds \right. \\ \left. + \left( \sum_{k=0}^{\infty} \rho^{-k} C^2 M^k \right) \|u - u^n\|_2^2 \right]. \end{aligned}$$

Since  $\beta_{00} \leq \beta$ , we then have for some  $K_2$

$$\begin{aligned} (9.5) \quad \beta(t) &\leq K_2 \left[ \int_0^t \beta(s) ds + \|u - u^n\|_2^2 \right], \\ \beta(t) &\leq \exp(K_2 T) \|u - u^n\|_2^2, \quad 0 \leq t \leq T. \end{aligned}$$

To complete the proof of Theorem 3, given  $(\pi, u, b)$  admissible (S) let  $u_t^n$ ,  $n = 1, 2, \dots$ , be a sequence of piecewise-constant controls such that  $\|u - u^n\|_2^2 \rightarrow 0$  as  $n \rightarrow \infty$ . The control  $u_t^n$  is constant on intervals  $[t_k^n, t_{k+1}^n)$  of some partition  $\Delta^n$  of  $[0, T]$ , and  $u_t^n$  is  $\mathcal{G}_{t_k^n}$  measurable on  $[t_k^n, t_{k+1}^n)$ . Since  $\mathcal{L}^0$  and  $\mathcal{L}^1$  are bounded operators, the technique of successive approximations provides a solution  $\pi_t^n$  to (4.2) corresponding to the initial data  $\pi_0$  and  $u_t^n, b_t$ . We omit the proof (one proof uses a method like the one above, with  $\varphi_t$  the difference between successive approximations to the solution  $\pi_t^n$ ). Inequality (9.5) implies that  $E\pi_t^n(g_j) \rightarrow E\pi_t(g_j)$  as  $n \rightarrow \infty$ . Since  $g_0, g_1, g_2, \dots$  span  $C(\Sigma)$ ,  $E\pi_t^n(g) \rightarrow E\pi_t(g)$  for all  $g \in C(\Sigma)$ ,  $0 \leq t \leq T$ . In particular,  $E\pi_T^n(\phi) \rightarrow E\pi_T(\phi)$ . Since  $\alpha_s \leq \alpha_s^{\Delta^n} \leq E\pi_T^n(\phi)$ ,

$$\alpha_s \leq \limsup_{n \rightarrow \infty} \alpha_s^{\Delta^n} \leq E\pi_T(\phi).$$

Since the infimum of the right side among all admissible  $(\pi, u, b)$  is  $\alpha_s$ , this proves Theorem 3.

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